CS 70 FALL 2007 — DISCUSSION #1

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1. Administrivia

(1) Course Information
- The first homework is **due tomorrow September 6** in 283 Soda Hall. You are encouraged to work on the homework in groups of 3-4, but write up your submission *on your own*. Cite any external sources you use.

(2) Discussion Information
- If you have a clash, it is OK to attend a section different to your enrolled/wait-listed one. Just be sure to show up so that we can 'assign' you somewhere based on the rolls taken in sections in the first few weeks.
- Section notes like these will be posted on the course website.
- Feel free to contact the GSI's via e-mail, or the class staff and students through the newsgroup, ucb.class.cs70, if you have a question.

2. A warm-up exercise

Exercise 1. Write down the truth table for $\neg A \to B$. What else is this operation on A and B known as?

3. Quantifier Practice

Consider the false statement "For each x in $\mathbb{R}$, $x^2 \geq x$" (consider $0 < x < 1$). What is the negation of this statement? Is it "For each x in $\mathbb{R}$, $x^2 < x$"? No, because this statement is still false (e.g. consider $x > 1$). So what is going wrong here?

Let $P(x)$ be the proposition "$x^2 \geq x$" with x taken from the universe of real numbers $\mathbb{R}$. Then our original statement is succinctly written as $\forall x, P(x)$. Using DeMorgan's laws, we get $\neg \forall x, P(x) \equiv \exists x, \neg P(x)$ or "There exists a real x for which $x^2 < x$."

We can chain together quantifiers in any manner we please: $\forall x, \exists y, \forall z, P(x, y, z)$ and negate it using the same rules discussed above. By applying the rules in sequence, we get that

$$\neg(\forall x.\exists y.\forall z.P(x, y, z))$$
$$\exists x.\neg(\exists y.\forall z.P(x, y, z))$$
$$\exists x.\forall y.\neg(\forall z.P(x, y, z))$$
$$\exists x.\forall y.\exists z, \neg P(x, y, z)$$

The $\neg$ "bubbles down", flipping quantifiers as it goes. The following problem comes from Question 14 in the Mathematics Subject GRE Sample Test:

Exercise 2. Let $\mathbb{R}$ be the set of real numbers and let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. The negation of the statement

"For each s in $\mathbb{R}$, there exists an r in $\mathbb{R}$ such that if $f(r) > 0$, then $g(s) > 0$."

is which of the following?

(A) For each s in $\mathbb{R}$, there exists an r in $\mathbb{R}$ such that $f(r) \leq 0$ and $g(s) > 0$.
(B) There exists an s in $\mathbb{R}$ such that for each r in $\mathbb{R}$, $f(r) \leq 0$ and $g(s) \leq 0$.
(C) There exists an s in $\mathbb{R}$ such that for each r in $\mathbb{R}$, $f(r) \leq 0$ and $g(s) > 0$.
(D) There exists an s in $\mathbb{R}$ such that for each r in $\mathbb{R}$, $f(r) > 0$ and $g(s) \leq 0$.
(E) For each s in $\mathbb{R}$, there exists an r in $\mathbb{R}$ such that $f(r) \leq 0$ and $g(s) \leq 0$.

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Use the tools covered above. (**hint:** what happens when you negate an implication? Try rewriting the statements in propositional logic, e.g. replacing $f(r) > 0$ with $P(r)$ and $g(s) > 0$ with $Q(s)$). $\square$

4. Bad Proofs

Exercise 3. Given $a, b \in \mathbb{R} - \{0\}$ and $ab > 1$, a student concludes $a > 1/b$. Is this always true? If not, where did the student go wrong? $\square$

5. Biconditional Proofs

Last thursday's lecture introduced a number of types of proofs, including direct proofs and proof by contraposition which both aim to prove a statement of the form $P \Rightarrow Q$. Often our goal will *additionally* be to prove the converse $Q \Rightarrow P$ – that is we are to prove $P \Leftrightarrow Q$.

Theorem 4. *n is odd iff n^2 is odd, for each $n \in \mathbb{N}$.*

Exercise 5. Consider Theorem 4.

(i) Begin by proving the forward direction (n odd implies n^2 odd). easy proof by algebra
(ii) Carefully prove the theorem with a simple modification to part (i).
(iii) Appeal to the equivalence of an implication and its contrapositive to prove the corollary[1] that "n is even iff n^2 is even, for each $n \in \mathbb{N}$.

$\square$

6. Different People, Different Proofs

Consider the following theorem.

Theorem 6. *Given a sequence of real numbers $x_0 = 1$ and $x_1, x_2, x_3, x_4, x_5 \geq 1$, the following holds true: if $x_5 > 35$, then $\exists i \in \{0, 1, 2, 3, 4\}$ such that $\frac{x_{i+1}}{x_i} > 2$.*

You can prove this in any of the three ways, you learnt in class: direct proof, proof by contrapositive and proof by contradiction.

Exercise 7. Prove the theorem in each of the three ways. Which one was easier? Which one more natural to you?

[1] A 'corollary' is a result that immediately follows from a proven result.