1. Expectation

**Exercise 1.** Suppose that I choose a permutation of the numbers 1, 2, ..., n uniformly at random. What is the expected number of entries that are greater than all preceding entries? For example, in the permutation 4, 2, 1, 5, 3, the numbers 4 and 5 are greater than all preceding entries (Hint: What’s the probability that the first entry is greater than all the preceding entries? What about the second one?)

2. Variance

**Exercise 2.** Suppose $X$ and $Y$ are independent. Find a formula for $\text{Var}(X - Y)$ in terms of $\text{Var}(X)$ and $\text{Var}(Y)$.

**Exercise 3.** The covariance, $\text{Cov}(X, Y)$, of two random variables, $X$ and $Y$, is defined to be $E(XY) - E(X)E(Y)$. Note that if two random variables are independent, then their covariance is zero.

1. Give an example to show that having $\text{Cov}(X, Y) = 0$ does not necessarily mean that $X$ and $Y$ are independent.
2. Let $X_1, \ldots, X_n$ be random variables. Prove that

$$\text{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j)$$

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3. **Binomial Distribution**

\[ X \sim Bin(n, p) : \Pr(X = k) = \binom{n}{k} p^k(1 - p)^{n-k} \]

**Exercise 4.** A rolls a die, and B rolls a die. A wins if the number showing on his die is strictly greater than the one on B's. If they play this game five times, what is the chance that A wins at least four times?

4. **Poisson Distribution**

\[ X \sim Poi(\lambda) : \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \]

**Exercise 5.** For a certain population, the average number of colds per year is distributed approximately Poisson with mean 2. Answer the following questions.

1. What is the probability that a person in this population will have four or fewer colds in the next year?
2. What is the probability that among 3 friends, one will have no colds, one exactly one cold, and one exactly two colds?

5. **Geometric Distribution**

\[ X \sim Geo(p) : \Pr(X = k) = (1 - p)^{k-1}p \]

**Exercise 6 (Generalized Geometric RV).** A coin having probability \( p \) of coming up heads is successively flipped until the \( r \)th head appears. Argue that \( X \), the number if flips required, will be \( n, n \geq r \), with probability

\[ P[X = n] = \binom{n-1}{r-1} p^r (1 - p)^{n-r} \]

Compute the expectation of \( X \).