CS 70 FALL 2007 — DISCUSSION #10

ASSANE GUEYE, LUQMAN HODGKINSON, AND VAHAB POURNAGHSHBAND

1. Expectation

Exercise 1. Suppose that I choose a permutation of the numbers $1, 2, ..., n$ uniformly at random. What is the expected number of entries that are greater than all preceding entries? For example, in the permutation 4, 2, 1, 5, 3, the numbers 4 and 5 are greater than all preceding entries (Hint: What's the probability that the first entry is greater than all the preceding entries? What about the second one?)

2. Variance

Exercise 2. Suppose X and Y are independent. Find a formula for $Var(X - Y)$ in terms of $Var(X)$ and $Var(Y)$.

Exercise 3. The covariance, $Cov(X, Y)$, of two random variables, X and Y , is defined to be $E(XY) - E(X)E(Y)$. Note that if two random variables are independent, then their covariance is zero.

(1) Give an example to show that having $Cov(X, Y) = 0$ does not necessarily mean that X and Y are independent.
(2) Let $X_1, \ldots, X_n$ be random variables. Prove that

$$Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{i<j} Cov(X_i, X_j)$$

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3. Binomial Distribution

$$X \sim Bin(n,p) : \Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Exercise 4. A rolls a die, and B rolls a die. A wins if the number showing on his die is strictly greater than the one on B's. If they play this game five times, what is the chance that A wins at least four times?

4. Poisson Distribution

$$X \sim Poi(\lambda) : \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Exercise 5. For a certain population, the average number of colds per year is distributed approximately Poisson with mean 2. Answer the following questions.

(1) What is the probability that a person in this population will have four or fewer colds in the next year?
(2) What is the probability that among 3 friends, one will have no colds, one exactly one cold, and one exactly two colds?

5. Geometric Distribution

$$X \sim Geo(p) : \Pr(X = k) = (1-p)^{k-1} p$$

Exercise 6 (Generalized Geometric RV)**.** A coin having probability p of coming up heads is successively flipped until the rth head appears. Argue that X, the number if flips required, will be n, $n \geq r$, with probability

$$P[X = n] = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

Compute the expectation of X.