CS 70 FALL 2007 — DISCUSSION #2

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1. Administrivia

(1) Course Information
- Reminder: The second homework is due September 13th at 5pm in 283 Soda Hall
- Mark your calendar:
  First Midterm: Wednesday 10/3, 6-8pm, in 10 Evans
  Second Midterm: Thursday 11/15, 7-9pm, in 10 Evans

(2) Discussion Information
- The first homework is graded and will be distributed in section.
- Section 105 (5-6pm) is very undersubscribed, so if you are in sections 101, 102, or 104, you are encouraged to switch to section 105 if your schedule permits.

2. Algebraic Inductions

Let's try some practice induction problems that look like those covered in lecture this week.

Exercise 1. Prove that $1^2 + 3^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$. □

Exercise 2. (i) A geometric series is an infinite sum of the form $1+x+x^2+x^3+x^4+\dots$ for some real x. Prove that the series' partial sum $1+x+x^2+\dots+x^n$ equals $\frac{x^{n+1}-1}{x-1}$. Many times a guess is good and then you can use induction to actually prove it.

(ii) An arithmetic series is a series of the form $\sum_{i=1}^{\infty} a_k$ where $a_{k+1} = a_k + d$ for each positive integer k and $a_1, d \in \mathbb{R}$ are picked arbitrarily. Find the closed-form partial sum of this series and prove your result by induction.

□

Exercise 3. Prove: $\prod_{i=2}^{n}(1-\frac{1}{i}) = \frac{1}{n}$ $(\forall n \in \mathbb{N})$. □

3. Strong Induction: Sums of Fibonacci & Recursion

Many of you may have heard of the Fibonacci sequence. We define $F_1 = 1, F_2 = 1$, and then define the rest of the sequence recursively: for $k \geq 3$, $F_k = F_{k-1} + F_{k-2}$. So the sequence ends up looking like:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

While not all positive integers are Fibonacci (e.g. 4), surprisingly we can express any positive integer as the sum of distinct terms in the Fibonacci sequence.

Date: September 11, 2007.

Theorem 1. *Every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.*

Exercise 4. Prove Theorem 1. □

Exercise 5. Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ and $a_0 = 1, a_1 = 2$.
Prove: $a_n \leq n + 2$ for all $n \geq 0$. □

Exercise 6. Stirling numbers
The Stirling number of the second kind, $S(n, k), \quad n, k \in \mathbf{N}$ is defined as the partition of $\{1, 2, \ldots, n\}$ into exactly k non-empty subsets with the convention $S(0, 0) = 1$ and $S(n, 0) = 0$ for all $n > 0$.

(1) Compute $S(n, k)$ for $k > n$, $S(n, n)$ for all n, $S(n, 1)$ for all n.
(2) Argue that $S(n, n-1) = \binom{n}{2}$, and $S(n, 2) = 2^{n-1} - 1$.
(3) Show that the Stirling number satisfy the recurrence equation $S(n, k) = S(n-1, k-1) + kS(n-1, k)$.
(4) Show that for all integer n and for all real number x
$$x^n = \sum_k S(n, k) x^{\underline{k}}$$
where $x^{\underline{k}}$ is defined such that $x^{\underline{k+1}} = x^{\underline{k+1}}(x - k)$.
Hint: note that $xx^{\underline{k+1}} = x^{\underline{k+1}} + kx^{\underline{k}}$.

□