CS 70 FALL 2007 — DISCUSSION #2

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1. Administrivia

(1) Course Information
- Reminder: The second homework is due September 13^{th} at 5pm in 283 Soda Hall
- Mark your calendar:
  First Midterm: Wednesday 10/3, 6-8pm, in 10 Evans
  Second Midterm: Thursday 11/15, 7-9pm, in 10 Evans

(2) Discussion Information
- The first homework is graded and will be distributed in section.
- Section 105 (5-6pm) is very undersubscribed, so if you are in sections 101, 102, or 104, you are encouraged to switch to section 105 if your schedule permits.

2. Algebraic Inductions

Let's try some practice induction problems that look like those covered in lecture this week.

Exercise 1. Prove that $1^2 + 3^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$. □

Exercise 2. (i) A geometric series is an infinite sum of the form $1+x+x^2+x^3+x^4+\dots$ for some real x. Prove that the series' partial sum $1+x+x^2+\dots+x^n$ equals $\frac{x^{n+1}-1}{x-1}$. Many times a guess is good and then you can use induction to actually prove it.

(ii) An arithmetic series is a series of the form $\sum_{i=1}^{\infty} a_k$ where $a_{k+1} = a_k + d$ for each positive integer k and $a_1, d \in \mathbb{R}$ are picked arbitrarily. Find the closed-form partial sum of this series and prove your result by induction.

□

Exercise 3. Prove: $\prod_{i=2}^{n}(1-\frac{1}{i}) = \frac{1}{n}$ $(\forall n \in \mathbb{N})$. □

3. Strong Induction: Sums of Fibonacci & Recursion

Many of you may have heard of the Fibonacci sequence. We define $F_1 = 1, F_2 = 1$, and then define the rest of the sequence recursively: for $k \geq 3$, $F_k = F_{k-1} + F_{k-2}$. So the sequence ends up looking like:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

While not all positive integers are Fibonacci (e.g. 4), surprisingly we can express any positive integer as the sum of distinct terms in the Fibonacci sequence.

Date: September 11, 2007.

Theorem 1. *Every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.*

Exercise 4. Prove Theorem 1. □

Proof. Let $P(n)$ be the statement that n can be expressed as the sum of distinct terms in the Fibonacci sequence. We begin with the base case $n = 1$. Since 1 is a term in the Fibonacci sequence (namely F_1), then $P(1)$ is true.

Now we proceed to the inductive step. We wish to show that $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$. So assume that $P(1), P(2), \ldots, P(n)$ hold. Now we consider $n+1$. There are two cases:

(1) $n+1$ is itself a Fibonacci number.
(2) $n+1$ is not a Fibonacci number.

If the former holds, then we're done. If the latter holds, then there must exist some positive integer k such that

$$F_k < n+1 < F_{k+1}.$$

Since $F_k < n+1$, we may decompose $n+1$ into $F_k + (n+1-F_k)$. But by definition, $(n+1-F_k) < n+1$ so by the inductive hypothesis we know that $P(n+1-F_k)$ is true, hence it may be expressed as such:

$$n+1-F_k = F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

where the subscripts are distinct.

Lemma 1.1. *$F_k, F_{i_1}, F_{i_2}, \cdots, F_{i_m}$ are distinct.*

Proof.

(1) $F_{i_1}, F_{i_2}, \cdots, F_{i_m}$ are distinct by the inductive hypothesis (i.e. $P(n+1-F_k)$ is true).
(2) $F_k \notin \{F_{i_1}, F_{i_2}, \cdots, F_{i_m}\}$
Proof by contradiction: Let $s = n+1-F_k$, so $n+1 = s + F_k$ where $s = F_{i_1} + F_{i_2} + \cdots + F_{i_m}$. We know $F_{k-1} + F_k = F_{k+1}$ and $F_{k-1} < F_k < F_{k+1}$ for $k > 2$; hence, $F_k + F_k = 2F_k > F_{k+1}$. Now assume $F_k \in \{F_{i_1}, F_{i_2}, \cdots, F_{i_m}\}$; therefore, $n+1 = 2F_k + \sum F_j$ which implies $F_k < n+1 < F_{k+1} < 2F_k < n+1$ and that is a contradition.

Therefore we have

$$n+1 = F_k + F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

and $P(n+1)$ holds. Thus by strong induction, $P(n)$ holds for all $n \geq 1$.

□

Exercise 5. Let the sequence $a_0, a_1, a_2, \ldots$ be defined by the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ and $a_0 = 1, a_1 = 2$.
Prove: $a_n \leq n+2$ for all $n \geq 0$. □

Proof. We prove the stronger claim that $a_n = n+1$. Let $P(n) =$ "$a_n = n+1$." Then we claim that $\forall n \in \mathbb{N}.\ P(n)$.
Proof by strong induction:

- Base cases: $P(0)$ and $P(1)$ are trivially true.

- Inductive step: We need to show that $P(n) \implies P(n+1)$ holds for all $n \geq 2$.
  We have

$$
\begin{aligned}
&\text{(by definition)} & a_{n+1} &= 2a_n - a_{n-1} \\
&\text{(by the inductive hypothesis)} & &= 2(n+1) - n \\
&\text{(by the distributive property, and simplification)} & &= n+2.
\end{aligned}
$$

Thus $\forall n \in \mathbb{N}.\ a_n = n+1$.
Now since $a_n = n+1$ and $n+1 \leq n+2$, $a_n \leq n+2$ by substitution. Hence, the original claim is true as well.

Notice how strengthening the claim actually made it *easier* to prove the theorem. Here is a case where it is easier to prove more than to prove less. This happens not infrequently with induction proofs, and it is a trick worth knowing about. □

Exercise 6. Stirling numbers
The Stirling number of the second kind, $S(n,k)$, $n,k \in \mathbf{N}$ is defined as the partition of $\{1,2,\ldots,n\}$ into exactly k non-empty subsets with the convention $S(0,0) = 1$ and $S(n,0) = 0$ for all $n > 0$.

(1) Compute $S(n,k)$ for $k > n$, $S(n,n)$ for all n, $S(n,1)$ for all n.
(2) Argue that $S(n,n-1) = \binom{n}{2}$, and $S(n,2) = 2^{n-1} - 1$.
(3) Show that the Stirling number satisfy the recurrence equation $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.
(4) Show that for all integer n and for all real number x

$$x^n = \sum_k S(n,k) x^{\underline{k}}$$

where $x^{\underline{k}}$ is defined such that $x^{\underline{k+1}} = x^{\underline{k+1}}(x-k)$.
Hint: note that $xx^{\underline{k+1}} = x^{\underline{k+1}} + kx^{\underline{k}}$.

□