CS 70 SPRING 2007 — DISCUSSION #2

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1. Administrivia

(1) Course Information
   - Reminder: The first homework is due January 30^{st} at 3pm in 283 Soda Hall

(2) Discussion Information
   - If you have a clash, it is OK to attend a section different to your enrolled/wait-listed one. Just be sure to show up so that we can 'assign' you somewhere based on the roles taken in sections in the first few weeks.

2. Algebraic Inductions

Let's try some practice induction problems that look like those covered in lecture this week.

Exercise 1. Prove that $1^2 + 3^2 + ... + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$. □

Exercise 2. (i) A geometric series is an infinite sum of the form $1 + x + x^2 + x^3 + x^4 + \ldots$ for some real x. Prove that the series' partial sum $1 + x + x^2 + \ldots + x^n$ equals $\frac{x^{n+1}-1}{x-1}$. Many times a guess is good and then you can use induction to actually prove it.

(ii) An arithmetic series is a series of the form $\sum_{i=1}^{\infty} a_k$ where $a_{k+1} = a_k + d$ for each positive integer k and $a_1, d \in \mathbb{R}$ are picked arbitrarily. Find the closed-form partial sum of this series and prove your result by induction.

□

3. Strong Induction: Sums of Fibonacci & Prime Numbers

Many of you may have heard of the Fibonacci sequence. We define $F_1 = 1, F_2 = 1$, and then define the rest of the sequence recursively: for $k \geq 3$, $F_k = F_{k-1} + F_{k-2}$. So the sequence ends up looking like:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

While not all positive integers are Fibonacci (e.g. 4), surprisingly we can express any positive integer as the sum of distinct terms in the Fibonacci sequence.

Theorem 1. *Every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.*

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Proof. Let $P(n)$ be the statement that n can be expressed as the sum of distinct terms in the Fibonacci sequence. We begin with the base case $n = 1$. Since 1 is a term in the Fibonacci sequence (namely F_1), then $P(1)$ is true.

Now we proceed to the inductive step. We wish to show that $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$. So assume that $P(1), P(2), \ldots, P(n)$ hold. Now we consider $n+1$. There are two cases:

(1) $n+1$ is itself a Fibonacci number.
(2) $n+1$ is not a Fibonacci number.

If the former holds, then we're done. If the latter holds, then there must exist some positive integer k such that

$$F_k < n+1 < F_{k+1}.$$

Since $F_k < n+1$, we may decompose $n+1$ into $F_k + (n+1-F_k)$. But by definition, $(n+1-F_k) < n+1$ so by the inductive hypothesis we know that $P(n+1-F_k)$ is true, hence it may be expressed as such:

$$n+1-F_k = F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

where the subscripts are distinct.

Lemma 1.1. *$F_k, F_{i_1}, F_{i_2}, \cdots, F_{i_m}$ are distinct.*

Proof.

(1) $F_{i_1}, F_{i_2}, \cdots, F_{i_m}$ are distinct by the inductive hypothesis (i.e. $P(n+1-F_k)$ is true).
(2) $F_k \notin \{F_{i_1}, F_{i_2}, \cdots, F_{i_m}\}$
Proof by contradiction: Let $s = n+1-F_k$, so $n+1 = s+F_k$ where $s = F_{i_1}+F_{i_2}+\cdots+F_{i_m}$. We know $F_{k-1}+F_k = F_{k+1}$ and $F_{k-1} < F_k < F_{k+1}$ for $k > 2$; hence, $F_k+F_k = 2F_k > F_{k+1}$. Now assume $F_k \in \{F_{i_1}, F_{i_2}, \cdots, F_{i_m}\}$; therefore, $n+1 = 2F_k + \sum F_j$ which implies $F_k < n+1 < F_{k+1} < 2F_k < n+1$ and that is a contradition.

Therefore we have

$$n+1 = F_k + F_{i_1} + F_{i_2} + \cdots + F_{i_m}$$

and $P(n+1)$ holds. Thus by strong induction, $P(n)$ holds for all $n \geq 1$.

□

Similarly one might attempt to prove the analogous result with primes (repeats allowed).

Exercise 3. Prove that all integers greater than one can be expressed as the product of primes. □