CS 70 SPRING 2007 — DISCUSSION #12

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1. Administrivia

(1) Course Information
   - Midterm Statistics: Mean: ∼30 Standard deviation: ∼10
   - Homework 9 is posted.

2. Binomial Distribution

$$X \sim Bin(n,p) : \Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Exercise 1. A rolls a die, and B rolls a die. A wins if the number showing on his die is strictly greater than the one on B's. If they play this game five times, what is the chance that A wins at least four times?

3. Poisson Distribution

$$X \sim Poi(\lambda) : \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Exercise 2. Suppose 1% of people in a large population are over 6 feet 3 inches tall. Approximately what is the chance that from a group of 200 people picked at random from this population, at least four people will be over 6 feet 3 inches tall?

4. Geometric Distribution

$$X \sim Geo(p) : \Pr(X = k) = (1-p)^{k-1} p$$

Exercise 3. James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces temporary amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability $\frac{1}{3}$.

(1) What is the probability of escaping after seven attempts?
(2) On the average, how long does it take before he realizes that the door is unlocked and escapes?
    Solution. Let T be the total number of attempts Mr. Bond must make before he escapes from his imprisonment. I.e. the first $T-1$ attempts are failures—where he either enters the AC duct or the sewer pipe—and the final attempt at trial T seems him enter the door. Thus T corresponds to the number of coin tosses landing tails until a head turns up, where a

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tail (Bond fails) comes up with probability 2/3 while a head (James Bond escapes) has probability 1/3. Hence $T \sim Geo(1/3)$.

Define X_t for $t \in \{1, 2, \ldots, T-1\}$ to be the time Bond spends on failed trial t. Then X the total time Bond spends trying to escape is a r.v. satisfying

$$X = \sum_{t=1}^{T-1} X_t$$

Clearly the failure trial times $X_1, \ldots, X_{T-1}$ are i.i.d. with distribution

$$X_t = \begin{cases} 2 & \text{with probability } 0.5 \\ 5 & \text{with probability } 0.5 \end{cases}$$

and so

$$\begin{aligned} \mathbb{E}[X_1] = \ldots &= \mathbb{E}[X_{T-1}] \\ &= 2 \cdot \Pr(X_t = 2) + 0.5 \cdot \Pr(X_t = 5) \\ &= 3.5 \ . \end{aligned}$$

Intuitively we can apply linearity of expectation to compute $\mathbb{E}[X]$. Instead of multiplying the summand expectation by a constant number of terms in the sum, we simply multiply by the expected number of terms $\mathbb{E}[T-1]$:

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\sum_{t=1}^{T-1} X_t\right] \\ &= \mathbb{E}[T-1]\,\mathbb{E}[X_t] \\ &= (\mathbb{E}[T]-1)\,\mathbb{E}[X_t] \\ &= (3-1)\cdot 3.5 \\ &= 7 \ . \end{aligned}$$